

# Position-dependent effective mass Schrödinger equations for PT-symmetric potentials

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We use the method of point canonical transformations and choose the Rosen-Morse-type potential as the reference potential to study exact solutions of the position-dependent effective mass Schrödinger equations. Choosing three position-dependent mass distributions, we construct seven exactly solvable target potentials with PT symmetry. The energy spectra of the bound states and corresponding wavefunctions for the PT-symmetric potentials are given in the exact closed forms. We also discuss the isospectrality of different Schrödinger equations with the same mass distribution or different mass distributions for different PT-symmetric potentials.

**KEY WORDS:** Schrödinger equation, position-dependent effective mass, PT symmetry

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## 1. Introduction

In [1], Bender and Boettcher proposed a conjecture that a PT-symmetric potential with non-Hermiticity has a real energy spectrum when PT symmetry is not spontaneously broken. A potential  $V(x)$  is said to possess PT symmetry if the relation  $V(-x) = V^*(x)$  or  $V(\xi - x) = V^*(x)$  exists under the transformation of  $x \rightarrow -x$  (or  $x \rightarrow \xi - x$ ) and  $i \rightarrow -i$ , where  $P$  denotes parity operator

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(space reflection) and  $T$  denotes time reversal. After the seminal work of Ref. [1], there has been considerable interest in studying the solutions of the Schrödinger equation and relativistic wave equations with a constant mass for non-Hermitian potentials with a real energy spectrum [2–15]. Non-Hermitian potential models have many applications in different research areas, such as nuclear physics [16], condensed matter [17], and population biology [18].

Recently, some authors [19–21] have studied the exact solvable Schrödinger equation with position-dependent mass for  $PT$ -symmetric potentials. Searching for the exact solutions of the Schrödinger equation and relativistic wave equation with the spatially dependent mass has attracted much attention [22–32]. The physical systems with position-dependent mass have been found to be very useful in the description of electronic properties of, for instance, compositionally graded crystals [33], quantum dots [34], and quantum liquids [35]. There are a few potentials, such as the smooth potential [22], Morse potential [23], and hard-core potential [24], one can be obtained the analytical solutions by solving directly the position-dependent effective mass Schrödinger equations. The popular procedure is to convert the Schrödinger equation with position-dependent mass to the Schrödinger equation with a constant mass by using various approaches. The point canonical transformation (PCT) has been proved to be very useful in the searching exact solutions of the Schrödinger equation with position-dependent mass [19, 25–27]. One can use the PCT and choose an exactly solvable potential as a reference potential to construct the exactly solvable target potential, which fall into the class that the Schrödinger equation with spatially dependent effective mass for some potentials can be solved exactly.

In this work, within the framework of the Schrödinger equation with spatially dependent mass, we use the PCT and choose the Rosen–Morse-type potential as the reference potential to construct some exactly solvable  $PT$ -symmetric potentials with a real energy spectrum.

## 2. Action of the PCT map on the Schrödinger equation

In this section, we present a brief account of the PCT approach in solving the Schrödinger equation with position-dependent mass [19, 25]. For a one-dimensional potential  $V(y)$ , the time independent Schrödinger wave equation with a constant mass is given by

$$\left( -\frac{1}{2} \frac{d^2}{dy^2} + V(y) \right) \Psi(y) = E \Psi(y), \quad (1)$$

where we have applied the atomic unit  $\hbar = 1$  and the constant mass  $M = 1$ . Making a transformation  $y \rightarrow x$  through a mapping function  $y = f(x)$ , and

performing the following transformation of the wavefunction  $\Psi(y)$ ,

$$\Psi(y) = g(x) \tilde{\Psi}(x), \quad (2)$$

we yield a transformed Schrödinger equation

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} - \left( \frac{g'}{g} - \frac{f''}{2f} \right) \frac{d}{dx} - \frac{1}{2} \left( \frac{g''}{g} - \frac{f''}{f'} \frac{g'}{g} \right) + (f')^2 V(f(x)) \right] \times \tilde{\Psi}(x) = (f')^2 E \tilde{\Psi}(x), \quad (3)$$

where the prime denotes differentiation with respect to the variable  $x$ . In the case of the symmetric ordering of the momentum and mass, the one-dimensional Schrödinger equation with position-dependent mass is given by

$$-\frac{1}{2} \frac{d}{dx} \left[ \frac{1}{M(x)} \frac{d\tilde{\Psi}(x)}{dx} \right] + \tilde{V}(x) \tilde{\Psi}(x) = \tilde{E} \tilde{\Psi}(x), \quad (4)$$

where  $M(x) = m_0 m(x)$ ,  $m(x)$  is the dimensionless mass distribution function. Putting  $m_0 = 1$ , equation (4) can be reduced to the form

$$\left( -\frac{1}{2} \frac{d^2}{dx^2} + \frac{m'}{2m} \frac{d}{dx} + m \tilde{V}(x) \right) \tilde{\Psi}(x) = m \tilde{E} \tilde{\Psi}(x). \quad (5)$$

Comparing equations (3) and (5), one observes that equation (3) is identical to equation (5) if the following conditions exist,

$$g(x) = \sqrt{f'/m}, \quad (6)$$

$$\tilde{V}(x) - \tilde{E} = \frac{(f')^2}{m} [V(f(x)) - E] - \frac{1}{2m} \left( \frac{g''}{g} - \frac{f''}{f'} \frac{g'}{g} \right). \quad (7)$$

We set  $f' = m^{1/2}$ . Substituting this expression into equations (6) and (7) leads us to obtain a new potential  $\tilde{V}(x)$ ,

$$\tilde{V}(x) = V(f(x)) + \frac{1}{8m} \left[ \frac{m''}{m} - \frac{7}{4} \left( \frac{m'}{m} \right)^2 \right]. \quad (8)$$

We call the old potential  $V(y)$  as the reference potential, and the new potential  $\tilde{V}(x)$  as the target potential  $V(y)$ . Expressing the bound-state energy spectrum and wavefunction of the reference potential as  $E_n$  and  $\Psi_n(y)$ , we obtain the energy spectrum  $\tilde{E}_n$  and wavefunction  $\tilde{\Psi}_n(x)$  of the target potential  $\tilde{V}(x)$  from equations (7) and (2),

$$\tilde{E}_n = E_n, \quad (9a)$$

$$\tilde{\Psi}_n(x) = \frac{1}{g(x)} \Psi_n(y) = m(x)^{1/4} \Psi_n(f(x)). \tag{9b}$$

In this work, we consider three kinds of position-dependent mass distributions and choose the Rosen–Morse-type potential as the reference potential to construct seven target potentials, which are  $PT$ -symmetric potentials with real energy spectra.

### 3. Examples

We take the Rosen–Morse-type potential [7] as the reference potential,

$$V(y) = -V_1 \operatorname{sech}_q^2 \alpha y - V_2 \tanh_q \alpha y, \tag{10}$$

where the parameter  $q$  is a real or complex parameter. Choosing appropriate parameters in this potential, it can turn to the Hermitian and non-Hermitian  $PT$ -symmetric versions of the Rosen–Morse well, Eckart potential, trigonometric form of the Rosen–Morse well, and trigonometric form of the Eckart potential [7, 8, 36]. The Rosen–Morse-type potential is a special case of the five-parameter exponential-type potential model [7]. In equation (10), we adopted the notations of the deformed hyperbolic functions introduced for the first time by Arai [37],

$$\begin{aligned} \sinh_q y &= \frac{e^y - qe^{-y}}{2}, & \cosh_q y &= \frac{e^y + qe^{-y}}{2}, & \operatorname{sech}_q y &= \frac{1}{\cosh_q y}, \\ \operatorname{cosech}_q y &= \frac{1}{\sinh_q y}, & \tanh_q y &= \frac{\sinh_q y}{\cosh_q y}, & \operatorname{coth}_q y &= \frac{\cosh_q y}{\sinh_q y}. \end{aligned} \tag{11}$$

With the help of the coordinate translation transformation, the deformed hyperbolic potentials can be reduced to the non-deformed hyperbolic potentials [38]. In [7], the authors obtained the analytical solution of the Schrödinger equation with a constant mass for the Rosen–Morse-type potential by using the function analysis method. The energy spectrum and the corresponding unnormalized wavefunction for the reference potential  $V(y)$  are given by, respectively [7].

$$E_n(n) = -\frac{V_2^2}{4\alpha^2} \frac{1}{\kappa^2} - \alpha^2 \kappa^2, \tag{12a}$$

$$\Psi_n(y) = (\cosh_q \alpha y)^n e^{-\frac{V_2}{2\alpha n} y} P_n^{-2p_+, -2p_-}(-\tanh_q \alpha y), \tag{12b}$$

where  $n = 0, 1, 2, \dots, < \sqrt{\frac{1}{4} + \frac{V_1}{q\alpha^2}} - \frac{1}{2}$ ,  $\kappa = n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{V_1}{q\alpha^2}}$ , and the parameters  $p_+$  and  $p_-$  are defined as,

$$p_{\pm} = \frac{1}{2} \left[ \kappa \pm \frac{V_2}{2\alpha^2} \frac{1}{\kappa} \right]. \tag{13}$$

Case 1. We consider the mass distribution as  $m(x) = (\gamma + x^2)^2 / (1 + x^2)^2$ , where  $\gamma$  is a real constant parameter. When  $\gamma = 1$ , the mass  $m(x)$  becomes a constant. This position-dependent mass distribution was investigated by some authors in [21, 25, 28, 29]. The mapping function is  $y = f(x) = \int m(x)^{1/2} dx = x + (\gamma - 1) \tan^{-1} x$ . Applying equation (8) and assuming  $q = 1$ ,  $V_1 = A(A + \alpha)$ , and  $V_2 = 2iB$  in equation (10), yields the target potential  $\tilde{V}_1(x)$ ,

$$\begin{aligned} \tilde{V}_1(x) = & -A(A + \alpha) \operatorname{sech}^2 \alpha \left( x + (\gamma - 1) \tan^{-1} x \right) \\ & - 2iB \tanh \alpha \left( x + (\gamma - 1) \tan^{-1} x \right) \\ & - \frac{\gamma - 1}{2(\gamma + x^2)^4} \left( 3x^4 + 2(2 - \gamma)x^2 - \gamma \right). \end{aligned} \tag{14}$$

For this potential, we can easily show  $\tilde{V}_1(-x) = \tilde{V}_1^*(x)$ . Hence, the target potential  $\tilde{V}_1(x)$  possesses PT symmetry.

Making the corresponding parameter replacements in equations (9), (12), and (13), the energy spectrum  $\tilde{E}_n$  and wavefunction  $\tilde{\Psi}_n(x)$  of the PT-symmetric potential  $\tilde{V}_1(x)$  are found to be, respectively,

$$\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n - A/\alpha)^2} - \alpha^2 (n - A/\alpha)^2, \tag{15a}$$

$$\begin{aligned} \tilde{\Psi}_n(x) = & \left( \frac{\gamma + x^2}{1 + x^2} \right)^{1/2} (\cosh \delta_1(x))^{(n - A/\alpha)} e^{-i \frac{B}{\alpha^2(n - A/\alpha)} \delta_1(x)} P_n^{-2p_+, -2p_-} \\ & \times (-\tanh(\delta_1(x))), \end{aligned} \tag{15b}$$

where  $n = 0, 1, 2, \dots, < A/\alpha$ , the function  $\delta_1(x)$  is defined as  $\delta_1(x) = \alpha(x + (\gamma - 1) \tan^{-1} x)$ , and the parameters  $p_+$  and  $p_-$  read, respectively,

$$p_+ = \frac{1}{2} \left[ n - A/\alpha + i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right], \quad p_- = \frac{1}{2} \left[ n - A/\alpha - i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right].$$

Case 2. The mass distribution is also still assumed as  $m(x) = (\gamma + x^2)^2 / (1 + x^2)^2$ . Making the parameter replacements of  $q = -e^{2i\alpha\varepsilon}$ ,  $V_1 = -A(A + \alpha)e^{2i\alpha\varepsilon}$ , and  $V_2 = 2iB$  in the Rosen–Morse-type potential (10), we can produce the target potential  $\tilde{V}_2(x)$ ,

$$\begin{aligned} \tilde{V}_2(x) = & A(A + \alpha) \operatorname{cosech}^2 \alpha \left( x + (\gamma - 1) \tan^{-1} x - i\varepsilon \right) \\ & - 2iB \operatorname{coth} \alpha \left( x + (\gamma - 1) \tan^{-1} x - i\varepsilon \right) \\ & - \frac{\gamma - 1}{2(\gamma + x^2)^4} \left( 3x^4 + 2(2 - \gamma)x^2 - \gamma \right), \end{aligned} \tag{16}$$

where  $0 < \varepsilon < \frac{\pi}{4}$  or  $-\frac{\pi}{4} < \varepsilon < 0$ . If we replace  $x$  by  $-x$ , we have

$$\begin{aligned} \cosh \alpha \left( x + (\gamma - 1) \tan^{-1} x - i\varepsilon \right) &\rightarrow \cosh \alpha \left( x + (\gamma - 1) \tan^{-1} x + i\varepsilon \right), \\ \sinh \alpha \left( x + (\gamma - 1) \tan^{-1} x - i\varepsilon \right) &\rightarrow -\sinh \alpha \left( x + (\gamma - 1) \tan^{-1} x + i\varepsilon \right). \end{aligned} \quad (17)$$

With the help of equation (17), we get  $\tilde{V}_2(-x) = \tilde{V}_2^*(x)$ , thus, the potential  $\tilde{V}_2(x)$  is also PT-symmetric.

After making the corresponding parameter replacements in equations (9), (12), and (13), the energy spectrum  $\tilde{E}_n$  and wavefunction  $\tilde{\Psi}_n(x)$  for the PT-symmetric potential  $\tilde{V}_2(x)$  are given by, respectively,

$$\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n - A/\alpha)^2} - \alpha^2 (n - A/\alpha)^2, \quad (18a)$$

$$\begin{aligned} \tilde{\Psi}_n(x) &= \left( \frac{\gamma + x^2}{1 + x^2} \right)^{1/2} \left( e^{i\alpha\varepsilon} \sinh \delta_2(x) \right)^{(n-A/\alpha)} e^{-i \frac{B}{\alpha^2(n-A/\alpha)} \delta_2(x)} P_n^{-2P_+, -2P_-} \\ &\times (-\coth(\delta_2(x))), \end{aligned} \quad (18b)$$

where  $n = 0, 1, 2, \dots, < A/\alpha$ , the function  $\delta_2(x)$  is defined as  $\delta_2(x) = \alpha(x + (\gamma - 1) \tan^{-1} x - i\varepsilon)$ , and the parameters  $p_+$  and  $p_-$  read, respectively,

$$p_+ = \frac{1}{2} \left[ n - A/\alpha + i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right], \quad p_- = \frac{1}{2} \left[ n - A/\alpha - i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right].$$

Comparing equations (15a) and (18a), we find that both PT-symmetric potentials  $\tilde{V}_1(x)$  and  $\tilde{V}_2(x)$  possess the same energy spectra. The corresponding two position-dependent effective mass Schrödinger equations have the same mass distribution and different potentials. Thus, these two different PT-symmetric potentials are isospectral.

In the following, we list some of the results obtained for the PT-symmetric target potentials constructed in this work.

- Case 3

Mass distribution:  $m(x) = \frac{(\gamma+x^2)^2}{(1+x^2)^2}$ .

Mapping function:  $y = f(x) = x + (\gamma - 1) \tan^{-1} x$ .

Parameter replacements:  $q = 1$ ,  $\alpha \rightarrow i\alpha$ ,  $V_1 = -A(A + \alpha)$ ,  $V_2 = -2B$ .

Target potential:  $\tilde{V}_3(x) = A(A + \alpha) \sec^2 \alpha f(x) + 2iB \tan \alpha f(x)$

$$-\frac{\gamma-1}{2(\gamma+x^2)^4} (3x^4 + 2(2-\gamma)x^2 - \gamma).$$

Energy spectrum:  $\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n-A/\alpha)^2} + \alpha^2 (n - A/\alpha)^2$ .

Wavefunction:

$$\tilde{\Psi}_n(x) = \left( \frac{\gamma + x^2}{1 + x^2} \right)^{1/2} (\cos \alpha f(x))^{(n-A/\alpha)} e^{-i \frac{B}{\alpha^2(n-A/\alpha)} \alpha f(x)} P_n^{-2P_+, -2P_-} \times (-i \tan(\alpha f(x))),$$

where  $n = 0, 1, 2, \dots, < A/\alpha$ ,  $p_+ = \frac{1}{2} \left[ n - A/\alpha + \frac{B}{\alpha^2} \frac{1}{(n-A/\alpha)} \right]$ ,

$$p_- = \frac{1}{2} \left[ n - A/\alpha - \frac{B}{\alpha^2} \frac{1}{(n-A/\alpha)} \right].$$

- Case 4

Mass distribution:  $m(x) = \frac{(\gamma+x^2)^4}{(1+x^2)^4}$ .

Mapping function:  $y = f(x) = x + \frac{(\gamma-1)^2}{2} \frac{x}{1+x^2} + \frac{1}{2} (\gamma - 1) (\gamma + 3) \tan^{-1} x$ .

Parameter replacements:  $q = 1$ ,  $\alpha \rightarrow i\alpha$ ,  $V_1 = -A(A + \alpha)$ ,  $V_2 = -2B$ ,

Target potential:  $\tilde{V}_4(x) = -A(A + \alpha) \sec^2 \alpha f(x) - 2iB \tanh \alpha f(x)$

$$-\frac{(\gamma-1)(1+x^2)^2}{(\gamma+x^2)^6} (-3x^4 + (5\gamma - 7)x^2 + \gamma).$$

Energy spectrum:  $\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n-A/\alpha)^2} - \alpha^2 (n - A/\alpha)^2$ .

Wavefunction:

$$\tilde{\Psi}_n(x) = \left( \frac{\gamma + x^2}{1 + x^2} \right)^2 (\cosh \alpha f(x))^{(n-A/\alpha)} e^{-i \frac{B}{\alpha^2(n-A/\alpha)} \alpha f(x)} P_n^{-2P_+, -2P_-} \times (-\tanh \alpha f(x)).$$

where  $n = 0, 1, 2, \dots, < A/\alpha$ ,

$$p_+ = \frac{1}{2} \left[ n - A/\alpha + i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right],$$

$$p_- = \frac{1}{2} \left[ n - A/\alpha - i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right].$$

• Case 5

Mass distribution:  $m(x) = \frac{(\gamma+x^2)^4}{(1+x^2)^4}$ .

Mapping function:  $y = f(x) = x + \frac{(\gamma-1)^2}{2} \frac{x}{1+x^2} + \frac{1}{2} (\gamma - 1) (\gamma + 3) \tan^{-1} x$ .

Parameter replacements:  $q = 1, \alpha \rightarrow i\alpha, V_1 = -A(A + \alpha) V_2 = -2B$ ,

Target potential:  $\tilde{V}_5(x) = A(A + \alpha) \sec^2 \alpha f(x) + 2iB \tan \alpha f(x)$

$$-\frac{(\gamma - 1) (1 + x^2)^2}{(\gamma + x^2)^6} (-3x^4 + (5\gamma - 7) x^2 + \gamma).$$

Energy spectrum:  $\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n-A/\alpha)^2} + \alpha^2 (n - A/\alpha)^2$ .

Wavefunction:

$$\tilde{\Psi}_n(x) = \left( \frac{\gamma + x^2}{1 + x^2} \right)^2 (\cos \alpha f(x))^{(n-A/\alpha)} e^{-i \frac{B}{\alpha^2(n-A/\alpha)} \alpha f(x)} P_n^{-2P_+, -2P_-} \times (-i \tan \alpha f(x)),$$

where  $n = 0, 1, 2, \dots, < A/\alpha, p_+ = \frac{1}{2} \left[ n - A/\alpha + \frac{B}{\alpha^2} \frac{1}{(n-A/\alpha)} \right]$ ,

$$p_- = \frac{1}{2} \left[ n - A/\alpha - \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right].$$



• Case 6

Mass distribution:  $m(x) = \frac{(\gamma+x^2)^4}{(1+x^2)^4}$ .

Mapping function:  $y = f(x) = x + \frac{(\gamma-1)^2}{2} \frac{x}{1+x^2} + \frac{1}{2} (\gamma - 1) (\gamma + 3) \tan^{-1} x$ .

Parameter replacements:  $q = -e^{2i\alpha\varepsilon}$ ,  $V_1 = -A(A + \alpha) e^{2i\alpha\varepsilon}$ ,  $V_2 = 2iB$ ,  
 where  $0 < \varepsilon < \frac{\pi}{4}$  or  $-\frac{\pi}{4} < \varepsilon < 0$

Target potential:  $\tilde{V}_6(x) = A(A + \alpha) \operatorname{cosech}^2 \alpha (f(x) - i\varepsilon) - 2iB \coth \alpha (f(x) - i\varepsilon) - \frac{(\gamma - 1)(1 + x^2)^2}{(\gamma + x^2)^6} (-3x^4 + (5\gamma - 7)x^2 + \gamma)$ .

Energy spectrum:  $\tilde{E}_n = \frac{B^2}{\alpha^2} \frac{1}{(n - A/\alpha)^2} - \alpha^2 (n - A/\alpha)^2$ .

Wavefunction:

$$\tilde{\Psi}_n(x) = \left( \frac{\gamma + x^2}{1 + x^2} \right)^2 \left( e^{i\alpha\varepsilon} \sinh(\alpha f(x) - i\alpha\varepsilon) \right)^{(n - A/\alpha)} e^{-i \frac{B}{\alpha^2(n - A/\alpha)} (\alpha f(x) - i\alpha\varepsilon)} \times P_n^{-2P_+, -2P_-} (-\coth(\alpha f(x) - i\alpha\varepsilon)),$$

where  $n = 0, 1, 2, \dots, < A/\alpha$ ,  $p_+ = \frac{1}{2} \left[ n - A/\alpha + i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right]$ ,

$$p_- = \frac{1}{2} \left[ n - A/\alpha - i \frac{B}{\alpha^2} \frac{1}{(n - A/\alpha)} \right].$$

• Case 7

Mass distribution:  $m(x) = \frac{a^2}{q+x^2}$ .

Mapping function:  $y = f(x) = a \ln \left[ x + \sqrt{q + x^2} \right]$ .

Parameter replacements:  $\alpha = 1/a$ ,  $q = 1$ ,  $V_1 = A(A + \alpha)$ ,  $V_2 = iB$ .

Target potential:  $\tilde{V}_7(x) = -\frac{1}{8a^2} - \frac{A^2 + A/a + 1}{1 + x^2} - iB \frac{x}{\sqrt{1 + x^2}}$ .

Energy spectrum:  $\tilde{E}_n = \frac{a^2 B^2}{4(n - aA)^2} - \frac{1}{a^2} (n - aA)^2$ .

Wavefunction:

$$\Psi_n(x) = \frac{\sqrt{a}}{(1+x^2)^{1/4}} \left( \cosh\left(\frac{1}{a}f(x)\right) \right)^{(n-aA)} e^{-i\frac{B}{2(n-aA)}af(x)} P_n^{-2p_+, -2p_-} \times \left( -\tanh\left(\frac{1}{a}f(x)\right) \right),$$

$$\text{where } n = 0, 1, 2, \dots, < aA, p_+ = \frac{1}{2} \left[ n - aA + i \frac{a^2 B}{2} \frac{1}{n - aA} \right],$$

$$p_- = \frac{1}{2} \left[ n - aA - i \frac{a^2 B}{2} \frac{1}{n - aA} \right].$$

#### 4. Conclusion

With the help of the PCT approach, we choose the Rosen–Morse-type potential as the reference potential to construct seven exact solvable  $PT$ -symmetric potentials, which fall into the class of position-dependent effective mass Schrödinger equations. In the ambience of the Schrödinger equation with position-dependent mass, the  $PT$ -symmetric potentials are non-Hermitian but have real energy spectra for the bound states. With the help of the energy spectrum formula and wavefunction expression for the Rosen–Morse-type potential within the framework of the standard Schrödinger equation with a constant mass, we obtain the energy levels of the bound states and corresponding wavefunctions for the seven  $PT$ -symmetric potentials in the exact closed forms. The two sets of  $PT$ -symmetric potentials  $\tilde{V}_1(x)$  and  $\tilde{V}_2(x)$ , and  $\tilde{V}_4(x)$  and  $\tilde{V}_6(x)$  are isospectral, and possess the same mass distribution in the position-dependent effective mass Schrödinger equations, respectively. The  $PT$ -symmetric potentials  $\tilde{V}_3(x)$  and  $\tilde{V}_5(x)$  are also isospectral, and possess the different mass distributions in the position-dependent effective mass Schrödinger equations.

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